

EXPONENTIAL BOUNDS FOR THE SUPPORT CONVERGENCE IN THE SINGLE RING THEOREM

FLORENT BENAYCH-GEORGES

ABSTRACT. We consider an $n \times n$ matrix of the form $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{V}$, with \mathbf{U}, \mathbf{V} some independent Haar-distributed unitary matrices and \mathbf{T} a deterministic matrix. We prove that for $k \sim n^{1/6}$ and $b^2 := \frac{1}{n} \text{Tr}(|\mathbf{T}|^2)$, as n tends to infinity, we have

$$\mathbb{E} \text{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) \lesssim b^{2k} \quad \text{and} \quad \mathbb{E}[|\text{Tr}(\mathbf{A}^k)|^2] \lesssim b^{2k}.$$

This gives a simple proof (with slightly weakened hypothesis) of the convergence of the support in the Single Ring Theorem, improves the available error bound for this convergence from $n^{-\alpha}$ to $e^{-cn^{1/6}}$ and proves that the rate of this convergence is at most $n^{-1/6} \log n$.

1. INTRODUCTION

The Single Ring Theorem, by Guionnet, Krishnapur and Zeitouni [8], describes the empirical distribution of the eigenvalues of a large generic matrix with prescribed singular values, *i.e.* an $n \times n$ matrix of the form $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{V}$, with \mathbf{U}, \mathbf{V} some independent Haar-distributed unitary matrices and \mathbf{T} a deterministic matrix whose singular values are the ones prescribed. More precisely, under some technical hypotheses¹, as the dimension n tends to infinity, if the empirical distribution of the singular values of \mathbf{A} converges to a compactly supported limit measure Θ on the real line, then the empirical eigenvalues distribution of \mathbf{A} converges to a limit measure μ on the complex plane which depends only on Θ . The limit measure μ (see Figure 1) is rotationally invariant in \mathbb{C} and its support is the annulus $\{z \in \mathbb{C}; a \leq |z| \leq b\}$, with $a, b \geq 0$ such that

$$a^{-2} = \int x^{-2} d\Theta(x) \quad \text{and} \quad b^2 = \int x^2 d\Theta(x). \quad (1)$$

In [9], Guionnet and Zeitouni also proved the convergence in probability of the support of the empirical eigenvalues distribution of \mathbf{A} to the support of μ . The reason why the radii

Date: March 11, 2015.

2000 Mathematics Subject Classification. 15B52; 60B20; 46L54.

Key words and phrases. Random matrices, Extreme eigenvalue statistics, Single Ring Theorem, Weingarten calculus, Haar measure, Free probability theory.

¹These hypotheses have been weakened by Rudelson and Vershynin in [19] and by Basak and Dembo in [1].

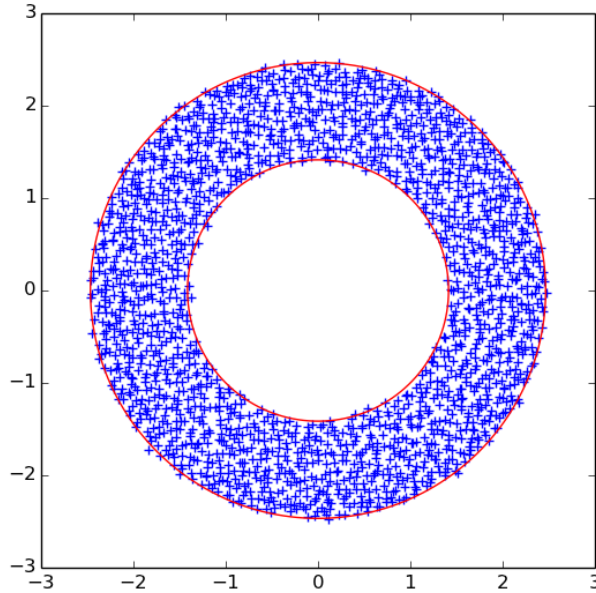


FIGURE 1. Spectrum of \mathbf{A} when the s_i 's are uniformly distributed on $[0.5, 4]$, so that $a \approx 1.41$ and $b \approx 2.47$ (here, $n = 2.10^3$).

a and b of the borders of the support of μ are given by (1) is related to the earlier work [10] by Haagerup and Larsen about R -diagonal elements in free probability theory but has no simple explanation: the matrix \mathbf{A} is far from being normal, hence its spectral radius should be smaller than its operator norm, *i.e.* than the L^∞ -norm² of Θ , but, up to our knowledge, there is no evidence why this modulus has to be close to the L^2 -norm of Θ , as follows from (1).

Another way to see the problem is the following one. In [19], Rudelson and Vershynin have proved that there is a universal constant c such that the smallest singular value $s_{\min}(z - \mathbf{A})$ of $z - \mathbf{A}$ has order at least n^{-c} as z varies in \mathbb{C} (and stays bounded away from 0 if \mathbf{A} is not invertible): this strong result seems incomplete, as it does not exhibit any transition as $|z|$ gets larger than b (by the Single Ring Theorem, we would expect a transition from the order n^{-c} to the order 1 as $|z|$ gets larger than b). Moreover, one cannot expect the methods of [19] to allow to prove such a transition, as they are based on the formula

$$s_{\min}(z - \mathbf{A}) \geq \frac{1}{\sqrt{n}} \times \min_{1 \leq i \leq n} \text{dist}_2(\textit{ith row of } z - \mathbf{A}, \text{span}(\text{other rows of } z - \mathbf{A})),$$

²To be precise, we should say the “ L^∞ -norm of a Θ -distributed *r.v.*” rather than “ L^∞ -norm of Θ ”. The same is true for the L^2 -norm hereafter.

whose RHT cannot have order larger than $n^{-1/2}$.

In this text, we want to fill in the gap of understanding why the borders of the support of μ have radii b and a (it suffices to understand the radius b , as a appears then naturally by considering \mathbf{A}^{-1} instead of \mathbf{A}). For this purpose, by an elementary moment expansion, we shed light on Formula (1) by proving that for $k \sim n^{1/6}$, the operator norm $\|\mathbf{A}^k\|$ has order at most b^k . More precisely, in Theorem 1, we show that

$$\mathbb{E} \operatorname{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) \lesssim b^{2k}. \quad (2)$$

This estimate allows to state some exponential bounds for the convergence of the moduli of the extreme eigenvalues of \mathbf{A} to a and b (see Corollary 2). As we said above, such a convergence had already been proved Guionnet and Zeitouni in [9] with some bounds of the type $n^{-\alpha}$ for an unspecified $\alpha > 0$, but in several applications (as the study of the outliers related to this matrix model in [2]), polynomial bounds are not enough, while exponential bounds are. We also slightly weaken the hypothesis for this convergence, using the paper [1] by Basak and Dembo. Then, the estimate (2) allows to give an upper-bound on the rate of convergence of the spectral radius $|\lambda_{\max}(\mathbf{A})|$ of \mathbf{A} to b as n tends to infinity: Corollary 3 states that

$$|\lambda_{\max}(\mathbf{A})| - b \lesssim n^{-1/6} \log n.$$

This result can be compared to the result of Rider in [15], who proved that the spectral radius of a Ginibre matrix fluctuates around its limit at rate $(n \log n)^{-1/2}$ (see also some generalizations in [3, 20, 4, 16]). At last, in Theorem 1, we also prove

$$\mathbb{E}[|\operatorname{Tr}(\mathbf{A}^k)|^2] \lesssim b^{2k} \quad (3)$$

with very little efforts, as the proof is mostly analogous to the one of (2). The estimate (3) is not needed to prove Corollaries 2 and 3, but will be of use in a forthcoming paper.

The main tools of the proofs are the so-called Weingarten calculus, an integration method for the Haar measure on the unitary group developed by Collins and Śniady in [5, 7], together with an exact formula for the Weingarten function (see (19)) proved by Mastomoto and Novak in their study of the relation between the Weingarten function and Jucys-Murphy elements in [11, 13]. A particularity of this paper is that the Weingarten calculus is used here to consider products of Haar-distributed unitary matrices entries with number of factors tending to infinity as the dimension n tends to infinity.

2. MAIN RESULTS

Let $n \geq 1$, $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{V}$ with $\mathbf{T} = \text{diag}(s_1, \dots, s_n)$ deterministic such that for all i , $s_i \geq 0$, and \mathbf{U}, \mathbf{V} some independent $n \times n$ Haar-distributed unitary matrices. Set

$$M := \max_{1 \leq i \leq n} s_i \quad \text{and} \quad b^2 := \frac{1}{n} \sum_{i=1}^n s_i^2.$$

Theorem 1. *Let $\varepsilon > 0$. There is a finite constant C depending only on ε (in particular, independent of n and of the s_i 's) such that for all positive integer k such that $k^6 < (2 - \varepsilon)n$, we have*

$$\mathbb{E} \text{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) \leq C n k^2 \left(b^2 + \frac{k M^2}{n} \right)^k \quad (4)$$

and

$$\mathbb{E}[|\text{Tr}(\mathbf{A}^k)|^2] \leq C \left(b^2 + \frac{k M^2}{n} \right)^k. \quad (5)$$

Let now $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ denote some eigenvalues of \mathbf{A} with respectively largest and smallest absolute values. We also introduce $a > 0$ defined by

$$\frac{1}{a^2} := \frac{1}{n} \sum_{i=1}^n s_i^{-2}$$

(with the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$). At last, for \mathbf{M} a matrix, $\|\mathbf{M}\|$ denotes the operator norm of \mathbf{M} with respect to the canonical Hermitian norm.

Corollary 2. *With the above notation, there are some constants $C, \delta_0 > 0$ depending only on M and b such that for any $\delta \in [0, \delta_0]$,*

$$\mathbb{P}(|\lambda_{\max}(\mathbf{A})| > b + \delta) \leq C n^{4/3} \exp \left(-\frac{n^{1/6} \delta}{C} \right) \quad (6)$$

and analogously, if $a > 0$ and $m := \min_i s_i > 0$, there are some constants $C, \delta_0 > 0$ depending only on a and m such that for any $\delta \in [0, \delta_0]$,

$$\mathbb{P}(|\lambda_{\min}(\mathbf{A})| < a - \delta) \leq C n^{4/3} \exp \left(-\frac{n^{1/6} \delta}{C} \right). \quad (7)$$

Proof. To prove (6), it suffices to notice that if $|\lambda_{\max}(\mathbf{A})| > b + \delta$, then $|\lambda_{\max}(\mathbf{A}^k)| > (b + \delta)^k$, which implies that $\|\mathbf{A}^k\| > (b + \delta)^k$ and that $\text{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) > (b + \delta)^{2k}$. Then the Tchebichev inequality and (4) allow to conclude. The proof of (7) then follows by application of (6) to \mathbf{A}^{-1} (the matrix \mathbf{A} is invertible as soon as $a > 0$). \square

The previous results are *non asymptotic* in the sense of [17, 18], meaning that they are true for all n (even though they involve some non specified constants). Let us now give two asymptotic corollaries. The proof of the first one follows directly from the previous lemma. Here, $u_n \gg v_n$ means $u_n/v_n \rightarrow 0$.

Corollary 3. *Let now the matrix $\mathbf{A} = \mathbf{A}_n$ depend on n and suppose that as n tends to infinity, the numbers $M = M_n$ and $b = b_n$ introduced in Theorem 1 stay bounded away from 0 and $+\infty$. Then for any sequence $\delta_n > 0$,*

$$\delta_n \gg n^{-1/6} \log n \implies \mathbb{P}(|\lambda_{\max}(\mathbf{A}_n)| > b_n + \delta_n) \xrightarrow{n \rightarrow \infty} 0.$$

The analogous result is also true for $\lambda_{\min}(\mathbf{A}_n)$.

Our last corollary allows a weakening of the hypotheses of the Single Ring Theorem support convergence proved in [9] by Guionnet and Zeitouni where we do not even have any single ring anymore. Let now the matrix $\mathbf{A} = \mathbf{A}_n$ depend on n , $\mathbf{T} = \mathbf{T}_n$ be random, independent from \mathbf{U}_n and \mathbf{V}_n and suppose that there is a (possibly random) $M_\infty > 1$ independent of n such that with probability tending to one, the spectrum of \mathbf{T} is contained in $[M_\infty^{-1}, M_\infty]$ and that there is a (possibly random) closed set $K \subset \mathbb{R}$ of zero Lebesgue measure such that for every $\varepsilon > 0$, there are some (possibly random) $\kappa_\varepsilon > 0$, M_ε and all n large enough,

$$\{z \in \mathbb{C}; \operatorname{Im}(z) > n^{-\kappa_\varepsilon}, \operatorname{Im}(\operatorname{Tr}((\mathbf{T} - z)^{-1}) > nM_\varepsilon\} \subset \cup_{x \in K} B(x, \varepsilon). \quad (8)$$

Corollary 4. • *If there is a finite (possibly random) number $b \geq 0$ such that as $n \rightarrow \infty$, we have the convergence in probability*

$$\frac{1}{n} \operatorname{Tr}(\mathbf{T}^2) \longrightarrow b^2,$$

then the spectral radius of \mathbf{A} converges in probability to b .

• *If there is a finite (possibly random) number $a > 0$ such that as $n \rightarrow \infty$, we have the convergences in probability*

$$\frac{1}{n} \operatorname{Tr}(\mathbf{T}^{-2}) \longrightarrow a^{-2},$$

then the minimal absolute value of the eigenvalues of \mathbf{A} converges in probability to a .

Proof. First of all, we shall only prove the first part of the corollary, the proof of the second one being an analogous consequence of Proposition 1.3 of [1]. Secondly, up to the replacement of \mathbf{T} by e.g. $\frac{M_\infty + M_\infty^{-1}}{2} I_n$ when its spectrum is not contained in $[M_\infty^{-1}, M_\infty]$ and to the conditioning with respect to the σ -algebra generated by the sequence $\{\mathbf{T}_n; n \geq 1\}$, one can suppose that \mathbf{T} is deterministic, as well as $M_\infty, b, K, \kappa_\varepsilon, M_\varepsilon$, that $\|\mathbf{T}^{-1}\|, \|\mathbf{T}\| \leq M_\infty$ and that

$$\frac{1}{n} \operatorname{Tr}(\mathbf{T}^2) \longrightarrow b.$$

Then as the set of probability measures supported by $[M_\infty^{-1}, M_\infty]$ is compact, up to an extraction, one can suppose that there is a probability measure Θ on $[M_\infty^{-1}, M_\infty]$ such that the empirical spectral law of \mathbf{T} converges to Θ as $n \rightarrow \infty$. It follows³, by Proposition 1.3 of [1], that within a subsequence, the empirical spectral law of \mathbf{A} converges in probability to a probability measure on \mathbb{C} whose support is a single ring, with maximal radius $\int x^2 d\Theta(x) = b^2$, hence that for any $\varepsilon > 0$,

$$\mathbb{P}(|\lambda_{\max}(\mathbf{A})| < b - \varepsilon) \longrightarrow 0.$$

By (6), the convergence in probability of the spectral radius to b is proved within a subsequence. In fact, we have even proved more: we have proved that from any subsequence, we can extract a subsequence within which the spectral radius of \mathbf{A} converges in probability to b . This is enough to conclude. \square

3. PROOF OF THEOREM 1

We first prove (4) (we will see below that the proof of (5) will go along the same lines, minus some border difficulties).

We have

$$\begin{aligned} \mathbb{E} \operatorname{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) &= \mathbb{E} \operatorname{Tr} \underbrace{\mathbf{U} \mathbf{T} \mathbf{V} \dots \mathbf{U} \mathbf{T} \mathbf{V}}_{k \text{ times}} \underbrace{\mathbf{V}^* \mathbf{T} \mathbf{U}^* \dots \mathbf{V}^* \mathbf{T} \mathbf{U}^*}_{k \text{ times}} \\ &= \mathbb{E} \operatorname{Tr} \underbrace{\mathbf{T} \mathbf{V} \mathbf{U} \dots \mathbf{T} \mathbf{V} \mathbf{U}}_{k-1 \text{ times}} \mathbf{T}^2 \underbrace{(\mathbf{V} \mathbf{U})^* \mathbf{T} \dots (\mathbf{V} \mathbf{U})^* \mathbf{T}}_{k-1 \text{ times}} \\ &= \mathbb{E} \operatorname{Tr} \underbrace{\mathbf{T} \mathbf{U} \dots \mathbf{T} \mathbf{U}}_{k-1 \text{ times}} \mathbf{T}^2 \underbrace{\mathbf{U}^* \mathbf{T} \dots \mathbf{U}^* \mathbf{T}}_{k-1 \text{ times}}, \end{aligned}$$

where we used the fact that $\mathbf{V} \mathbf{U} \stackrel{\text{law}}{=} \mathbf{U}$.

Let us denote $\mathbf{U} = [u_{ij}]_{i,j=1}^n$ and $\mathbf{U}^* = [u_{ij}^*]_{i,j=1}^n$. Then continuing the previous computation, we get

$$\begin{aligned} \mathbb{E} \operatorname{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) &= \\ \sum_{\substack{\mathbf{i}=(i_1, \dots, i_k) \\ \mathbf{j}=(j_1, \dots, j_k) \\ j_1=i_1, j_k=i_k}} \mathbb{E} s_{i_1} u_{i_1 i_2} s_{i_2} u_{i_2 i_3} \dots s_{i_{k-1}} u_{i_{k-1} i_k} s_{i_k} s_{j_k} u_{j_k j_{k-1}}^* s_{j_{k-1}} u_{j_{k-1} j_{k-2}}^* s_{j_{k-2}} \dots u_{j_2 j_1}^* s_{j_1}. \end{aligned}$$

³In Proposition 1.3 of [1] there is the supplementary hypothesis that Θ is not a Dirac mass, but this restriction might be there only for the harmonic analysis characterization of the limit measure true. Indeed, if $\Theta = \delta_b$, then the convergence of the empirical spectral law of \mathbf{A} to the uniform measure on the circle with radius b is obvious from the convergence of the empirical spectral distribution of a Haar-distributed unitary matrix to the uniform law on the unit circle and from classical perturbation inequalities.

By left and right invariance of the Haar measure (see Proposition 8), for the expectation in the RHT to be non zero, we need to have the equality of multisets

$$\{j_1, \dots, j_k\}_m = \{i_1, \dots, i_k\}_m$$

(the subscript m is used to denote multisets here). So

$$\mathbb{E} \operatorname{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) =$$

$$\begin{aligned} & \sum_{\mathbf{i}=(i_1, \dots, i_k)} \sum_{\substack{\mathbf{j}=(j_1, \dots, j_k) \\ j_1=i_1, j_k=i_k \\ \{j_2, \dots, j_{k-1}\}_m = \{i_2, \dots, i_{k-1}\}_m}} s_{i_1} \cdots s_{i_k} s_{j_k} \cdots s_{j_1} \mathbb{E} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_{k-1} i_k} u_{j_k j_{k-1}}^* s_{j_{k-1}} u_{j_{k-1} j_{k-2}}^* \cdots u_{j_2 j_1}^* \\ &= \sum_{\mathbf{i}=(i_1, \dots, i_k)} \sum_{\substack{\mathbf{j}=(j_1, \dots, j_k) \\ j_1=i_1, j_k=i_k \\ \{j_2, \dots, j_{k-1}\}_m = \{i_2, \dots, i_{k-1}\}_m}} s_{i_1}^2 \cdots s_{i_k}^2 \mathbb{E} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_{k-1} i_k} u_{j_k j_{k-1}}^* u_{j_{k-1} j_{k-2}}^* \cdots u_{j_2 j_1}^* . \end{aligned}$$

Let us define the subgroup \mathfrak{S}_k^0 of the k th symmetric group \mathfrak{S}_k by

$$\mathfrak{S}_k^0 := \{\varphi \in \mathfrak{S}_k; \varphi(1) = 1, \varphi(k) = k\},$$

and for each $\mathbf{i} = (i_1, \dots, i_k)$, we define the stabiliser group

$$\mathfrak{S}_k^0(\mathbf{i}) := \{\alpha \in \mathfrak{S}_k^0; \forall \ell, i_{\alpha(\ell)} = i_\ell\}.$$

Let at last $\mathfrak{S}_k^0 / \mathfrak{S}_k^0(\mathbf{i})$ denote the quotient of the set \mathfrak{S}_k^0 by $\mathfrak{S}_k^0(\mathbf{i})$ for the left action

$$(\alpha, \varphi) \in \mathfrak{S}_k^0(\mathbf{i}) \times \mathfrak{S}_k^0 \longmapsto \alpha \varphi.$$

Remark that the notation $i_{\Phi(1)}, \dots, i_{\Phi(k)}$ makes sense for $\Phi \in \mathfrak{S}_k^0 / \mathfrak{S}_k^0(\mathbf{i})$ even though Φ is not a permutation but a set of permutations. Then we have

$$\mathbb{E} \operatorname{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) = \sum_{\mathbf{i}=(i_1, \dots, i_k) \in \{1, \dots, n\}^k} (s_{i_1}^2 \cdots s_{i_k}^2 \times F_{\mathbf{i}}) \quad (9)$$

with

$$F_{\mathbf{i}} := \sum_{\Phi \in \mathfrak{S}_k^0 / \mathfrak{S}_k^0(\mathbf{i})} \mathbb{E} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_{k-1} i_k} u_{i_{\Phi(k)} i_{\Phi(k-1)}}^* \cdots u_{i_{\Phi(2)} i_{\Phi(1)}}^* . \quad (10)$$

Let us apply Proposition 8 to compute the expectation in the term associated to an element $\Phi \in \mathfrak{S}_k^0$. Let Wg denote the Weingarten function, introduced in Proposition 8. Let $c \in \mathfrak{S}_k$ be the cycle $(12 \cdots k)$. For any $\mathbf{i} = (i_1, \dots, i_k)$ and any $\Phi \in \mathfrak{S}_k^0$,

$$\begin{aligned} & \mathbb{E} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_{k-1} i_k} u_{i_{\Phi(k)} i_{\Phi(k-1)}}^* \cdots u_{i_{\Phi(2)} i_{\Phi(1)}}^* = \\ & \mathbb{E} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_{k-1} i_k} \overline{u_{i_{\Phi(1)} i_{\Phi(2)}}} \cdots \overline{u_{i_{\Phi(k-1)} i_{\Phi(k)}}} = \\ & \sum_{\sigma, \tau \in \mathfrak{S}_{k-1}} \delta_{i_1, i_{\Phi \sigma(1)}} \cdots \delta_{i_{k-1}, i_{\Phi \sigma(k-1)}} \delta_{i_2, i_{\Phi c \tau c^{-1}(2)}} \cdots \delta_{i_k, i_{\Phi c \tau c^{-1}(k)}} \operatorname{Wg}(\sigma^{-1} \tau) . \end{aligned}$$

But for any $\Phi \in \mathfrak{S}_k^0$ and any $\sigma, \tau \in \mathfrak{S}_{k-1}$, with $\bar{\sigma}, \bar{\tau} \in \mathfrak{S}_k$ defined by

$$\bar{\sigma}(x) = \begin{cases} \sigma(x) & \text{if } 1 \leq x \leq k-1, \\ k & \text{if } x = k, \end{cases} \quad \bar{\tau}(x) = \begin{cases} \tau(x) & \text{if } 1 \leq x \leq k-1, \\ k & \text{if } x = k, \end{cases}$$

we have (with the conventions that (xy) denotes the transposition of x and y when $x \neq y$ and the identity otherwise and that for $\pi \in \mathfrak{S}_k$ such that $\pi(k) = k$, $\pi|_{\{1, \dots, k-1\}}$ denotes the restriction of π to $\{1, \dots, k-1\}$)

$$\begin{aligned} i_1 = i_{\Phi\sigma(1)}, \dots, i_{k-1} = i_{\Phi\sigma(k-1)} &\iff i_1 = i_{\Phi\bar{\sigma}(1)}, \dots, i_k = i_{\Phi\bar{\sigma}(k)} \\ &\iff \text{for } \ell_1 = \sigma^{-1}(1), \forall \varphi \in \Phi, \varphi\bar{\sigma}(1 \ell_1) \in \mathfrak{S}_k^0(\mathbf{i}) \\ &\iff \exists \ell_1 \in \{1, \dots, k-1\}, \forall \varphi \in \Phi, \varphi\bar{\sigma}(1 \ell_1) \in \mathfrak{S}_k^0(\mathbf{i}) \\ &\iff \exists \ell_1 \in \{1, \dots, k-1\}, \exists \varphi_1 \in \Phi, \sigma = (\varphi_1^{-1}(1 \ell_1))|_{\{1, \dots, k-1\}} \end{aligned}$$

and

$$\begin{aligned} i_2 = i_{\Phi c\tau c^{-1}(2)}, \dots, i_k = i_{\Phi c\tau c^{-1}(k)} &\iff i_1 = i_{\Phi c\bar{\tau} c^{-1}(1)}, \dots, i_k = i_{\Phi c\bar{\tau} c^{-1}(k)} \\ &\iff \text{for } \ell_2 = c\tau^{-1}(k-1), \forall \varphi \in \Phi, \\ &\quad \varphi c\bar{\tau} c^{-1}(\ell_2 k) \in \mathfrak{S}_k^0(\mathbf{i}) \\ &\iff \exists \ell_2 \in \{2, \dots, k\}, \forall \varphi \in \Phi, \\ &\quad \varphi c\bar{\tau} c^{-1}(\ell_2 k) \in \mathfrak{S}_k^0(\mathbf{i}) \\ &\iff \exists \ell_2 \in \{2, \dots, k\}, \exists \varphi_2 \in \Phi, \\ &\quad c\bar{\tau} c^{-1} = \varphi_2^{-1}(\ell_2 k) \\ &\iff \exists \ell_2 \in \{1, \dots, k-1\}, \exists \varphi_2 \in \Phi, \\ &\quad \bar{\tau} = c^{-1}\varphi_2^{-1}c(\ell_2 k-1) \\ &\iff \exists \ell_2 \in \{1, \dots, k-1\}, \exists \varphi_2 \in \Phi, \\ &\quad \tau = (c^{-1}\varphi_2^{-1}c(\ell_2 k-1))|_{\{1, \dots, k-1\}}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_k i_{k+1}} u_{i_{\Phi(k)} i_{\Phi(k)}}^* \cdots u_{i_{\Phi(2)} i_{\Phi(1)}}^* &= \\ \sum_{\substack{(\varphi_1, \varphi_2) \in \Phi \times \Phi \\ 1 \leq \ell_1, \ell_2 \leq k-1}} \text{Wg}(((1 \ell_1) \varphi_1 c^{-1} \varphi_2^{-1} c(\ell_2 k-1))|_{\{1, \dots, k-1\}}) &= \\ \sum_{\substack{(\varphi_1, \varphi_2) \in \Phi \times \Phi \\ 1 \leq \ell_1, \ell_2 \leq k-1}} \text{Wg}((c^{-1} \varphi_2^{-1} c(\ell_2 k-1)(1 \ell_1) \varphi_1)|_{\{1, \dots, k-1\}}), & \end{aligned}$$

where we used the fact that Wg is a central function.

Thus by the definition of F_i at (10), we have

$$\begin{aligned}
F_i &= \sum_{\Phi \in \mathfrak{S}_k^0 / \mathfrak{S}_k^0(\mathbf{i})} \sum_{\substack{(\varphi_1, \varphi_2) \in \Phi \times \Phi \\ 1 \leq \ell_1, \ell_2 \leq k-1}} \text{Wg}((c^{-1} \varphi_2^{-1} c (\ell_2 k - 1) (1 \ell_1) \varphi_1)_{|\{1, \dots, k-1\}}) \\
&= \sum_{1 \leq \ell_1, \ell_2 \leq k-1} \sum_{\substack{(\varphi_1, \varphi_2) \in \mathfrak{S}_k^0 \times \mathfrak{S}_k^0 \\ \varphi_1 = \varphi_2 \text{ in } \mathfrak{S}_k^0 / \mathfrak{S}_k^0(\mathbf{i})}} \text{Wg}((c^{-1} \varphi_2^{-1} c (\ell_2 k - 1) (1 \ell_1) \varphi_1)_{|\{1, \dots, k-1\}}) \\
&= \sum_{1 \leq \ell_1, \ell_2 \leq k-1} \sum_{(\varphi, \alpha) \in \mathfrak{S}_k^0 \times \mathfrak{S}_k^0(\mathbf{i})} \text{Wg}((c^{-1} \varphi^{-1} \alpha^{-1} c (\ell_2 k - 1) (1 \ell_1) \varphi)_{|\{1, \dots, k-1\}}). \quad (11)
\end{aligned}$$

To state the following lemma, we need to introduce some notation: for $\sigma \in \mathfrak{S}_k$, let $|\sigma|$ denote the minimal number of factors necessary to write σ as a product of transpositions.

Lemma 5. *Suppose that $k^2 < 2n$. Then for any $\pi \in \mathfrak{S}_k \setminus \{id\}$, we have*

$$|\text{Wg}(\pi)| \leq \frac{2}{n^k k^2} \left(\frac{k^2}{2n} \right)^{|\pi|} \frac{1}{1 - \frac{k^2}{2n}} \quad (12)$$

and

$$|\text{Wg}(id)| \leq \frac{1}{n^k} + \frac{k^2}{2n^{k+2}} \frac{1}{1 - \frac{k^4}{4n^2}}. \quad (13)$$

Proof. We know, by (19), that the (implicitly depending on n) function Wg can be written

$$\text{Wg}(\pi) = \frac{1}{n^k} \sum_{r \geq 0} (-1)^r \frac{c_r(\pi)}{n^r},$$

with

$$c_r(\pi) := \#\{(s_1, \dots, s_r, t_1, \dots, t_r); \forall i, 1 \leq s_i < t_i \leq k, t_1 \leq \dots \leq t_r, \pi = (s_1 t_1) \cdots (s_r t_r)\}.$$

But for $r \geq 1$,

$$c_r(\pi) \leq \mathbb{1}_{r \geq |\pi|} \binom{k}{2}^{r-1} \leq \mathbb{1}_{r \geq |\pi|} \frac{k^{2r-2}}{2^{r-1}},$$

so that for $\pi \neq id$,

$$|\text{Wg}(\pi)| \leq \frac{2}{n^k k^2} \sum_{r \geq |\pi|} \left(\frac{k^2}{2n} \right)^r \leq \frac{2}{n^k k^2} \left(\frac{k^2}{2n} \right)^{|\pi|} \frac{1}{1 - \frac{k^2}{2n}},$$

whereas

$$|\text{Wg}(id)| \leq \frac{1}{n^k} + \frac{1}{n^k} \sum_{r \geq 1} \frac{|c_{2r}(id)|}{n^{2r}} \leq \frac{1}{n^k} + \frac{2}{n^k k^2} \sum_{r \geq 1} \left(\frac{k^2}{2n} \right)^{2r} = \frac{1}{n^k} + \frac{k^2}{2n^{k+2}} \frac{1}{1 - \frac{k^4}{4n^2}}.$$

□

Remark 6. Some other upper-bounds have been given for the Weingarten function: Theorem 4.1 of [6] and Lemma 16 of [12]. The first one states that for any $k, j \geq 2$ such that $k^j \leq n$, there is K_j depending only on j such that for all $\pi \in \mathfrak{S}_k$,

$$|\text{Wg}(\pi)| \leq K_j n^{-k-|\pi|(1-2/j)}, \quad (14)$$

whereas the second one states that if $k^{3/2} \leq n$, then for all $\pi \in \mathfrak{S}_k$,

$$|\text{Wg}(\pi)| \leq \frac{3C_{k-1}}{2} n^{-k-|\pi|}, \quad (15)$$

where C_{k-1} is the Catalan number of index $k-1$. However, in our case, these bounds are less relevant than the one we give here. Indeed, (15) allows to weaken the hypothesis $k^6 \leq (2-\varepsilon)n$ to $k^4 \leq (2-\varepsilon)n$, but, because of the Catalan number, contains implicitly a factor 4^k , which would change our main result from

$$\mathbb{E} \text{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) \lesssim b^{2k}$$

to

$$\mathbb{E} \text{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) \lesssim (2b)^{2k}$$

(which is far less interesting in our point of view, as explained in the introduction). On the other hand, (14) allows to turn the hypothesis $k^6 \leq (2-\varepsilon)n$ to

$$k^{\max\{j, 4j/(j-2)\}} \leq (2-\varepsilon)n,$$

but there is no integer j making it a good deal.

It follows from Lemma 5 (that we apply for $k-1$ instead of k) and from (11) that

$$\begin{aligned} |F_i| \leq & \sum_{1 \leq \ell_1, \ell_2 \leq k-1} \sum_{\alpha \in \mathfrak{S}_k^0(i)} \left\{ \right. \\ & \# \{ \varphi \in \mathfrak{S}_k^0; c^{-1} \varphi^{-1} \alpha^{-1} c (\ell_2 k - 1) (1 \ell_1) \varphi = id \} \left(\frac{1}{n^{k-1}} + \frac{k^2}{2n^{k+1}} \frac{1}{1 - \frac{k^4}{4n^2}} \right) + \\ & \frac{1}{1 - (k^2/(2n))} \sum_{q=1}^{k-2} \left(\# \{ \varphi \in \mathfrak{S}_k^0; |c^{-1} \varphi^{-1} \alpha^{-1} c (\ell_2 k - 1) (1 \ell_1) \varphi| = q \} \times \right. \\ & \left. \left. \frac{2}{n^{k-1} k^2} \left(\frac{k^2}{2n} \right)^q \right) \right\} \quad (16) \end{aligned}$$

(note that we suppress the restriction to the set $\{1, \dots, k-1\}$ because adding a fixed point does not change the minimal number of transposition needed to write a permutation).

Thus to upper-bound $|F_1|$, we need to upper-bound, for $1 \leq \ell_1, \ell_2 \leq k-1$, $\alpha \in \mathfrak{S}_k^0(\mathbf{i})$ and $q \in \{0, \dots, k-2\}$ fixed, the cardinality of the set of φ 's in \mathfrak{S}_k^0 such that

$$|c^{-1}\varphi^{-1}\alpha^{-1}c(\ell_2 k - 1)(1 \ell_1)\varphi| = q.$$

Lemma 7. *Let $q \in \{0, \dots, k-2\}$, $1 \leq \ell_1, \ell_2 \leq k-1$ and $\alpha \in \mathfrak{S}_k^0$ be fixed. Then we have*

$$\#\{\varphi \in \mathfrak{S}_k^0; |c^{-1}\varphi^{-1}\alpha^{-1}c(\ell_2 k - 1)(1 \ell_1)\varphi| = q\} \leq \frac{k^{4q}}{(2q)!}. \quad (17)$$

Proof. Let $\varphi \in \mathfrak{S}_k^0$ and define

$$\pi_\varphi := c^{-1}\varphi^{-1}\alpha^{-1}c(\ell_2 k - 1)(1 \ell_1)\varphi.$$

Note that for any $x \in \{1, \dots, k\}$,

$$\pi_\varphi(x) = x \implies \varphi(x) = (1 \ell_1)(\ell_2 k - 1)c^{-1}\alpha\varphi c(x),$$

so that $\varphi(x)$ is determined by $\varphi(c(x))$ (and by ℓ_1, ℓ_2 and α , which are considered as fixed here).

- It first follows that if $|\pi_\varphi| = 0$, *i.e.* if all x 's are fixed points of π_φ , then as by definition of \mathfrak{S}_k^0 , we always have $\varphi(k) = k$, φ is entirely defined by α, ℓ_1 and ℓ_2 . It proves (17) for $q = 0$.

- (i) If $|\pi_\varphi| \neq 0$, *i.e.* if π_φ has not only fixed points, then it follows that the values of φ on the complementary of the support of π_φ are entirely determined by its values on the support of π_φ (and by ℓ_1, ℓ_2 and α).

(ii) Let us now define, for each $\sigma \in \mathfrak{S}_k$ such that $|\sigma| = q$, a subset $\mathbf{A}(\sigma)$ of $\{1, \dots, k\}$ such that

$$\#\mathbf{A}(\sigma) = 2q \quad \text{and} \quad \text{supp}(\sigma) \subset \mathbf{A}(\sigma).$$

Such a set can be defined as follows: any permutation σ at distance q from the identity admits one (and only one, in fact, but we do not need this here) factorization

$$\sigma = (s_1 t_1) \cdots (s_q t_q)$$

such that $s_i < t_i$ and $t_1 < \cdots < t_q$. One can choose $\mathbf{A}(\sigma)$ to be $\{s_1, t_1, \dots, s_q, t_q\}$, possibly arbitrarily completed to a set with cardinality $2q$ if needed.

(iii) By what precedes, the map

$$\begin{aligned} \{\varphi \in \mathfrak{S}_k^0; |\pi_\varphi| = q\} &\longrightarrow \bigcup_{\substack{A \subset \{1, \dots, k\} \\ \#A = 2q}} \{1, \dots, k\}^A \\ \varphi &\longmapsto \varphi|_{\mathbf{A}(\pi_\varphi)} \end{aligned}$$

is one-to-one, and

$$\#\{\varphi \in \mathfrak{S}_k^0; |\pi_\varphi| = q\} \leq \binom{k}{2q} k^{2q} \leq \frac{k^{4q}}{(2q)!}.$$

□

It follows from this lemma and from (16) that for a constant C depending only on the ε of the statement of the theorem (this constant might change from line to line),

$$|F_{\mathbf{i}}| \leq k^2 \# \mathfrak{S}_k^0(\mathbf{i}) \left\{ \frac{1}{n^{k-1}} + \frac{Ck^2}{n^{k+1}} + \frac{C}{n^{k-1}k^2} \sum_{q=1}^{k-1} \frac{1}{(2q)!} \left(\frac{k^6}{2n} \right)^q \right\} \leq \frac{Ck^2}{n^{k-1}} \# \mathfrak{S}_k^0(\mathbf{i}).$$

Thus by (9),

$$\mathbb{E} \operatorname{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) \leq \frac{Ck^2}{n^{k-1}} \sum_{\mathbf{i}=(i_1, \dots, i_k) \in \{1, \dots, n\}^k} (s_{i_1}^2 \cdots s_{i_k}^2 \times \# \mathfrak{S}_k^0(\mathbf{i})).$$

Let us rewrite the last sum as follows:

- we first choose the number $p \in \{1, \dots, k\}$ such that $\#\{i_1, \dots, i_k\} = p$,
- then we choose the set $\{i_1^0 < \dots < i_p^0\} \subset \{1, \dots, n\}$ such that we have the equality of sets $\{i_1, \dots, i_k\} = \{i_1^0, \dots, i_p^0\}$ (we have $\binom{n}{p}$ possibilities),
- then we choose the collection $(\lambda_1, \dots, \lambda_p)$ of positive integers summing up to k such that in (i_1, \dots, i_k) , i_1^0 appears λ_1 times, \dots , i_p^0 appears λ_p times (we have $\binom{k-1}{p-1}$ possibilities),
- at last we choose a collection $\mathbf{S} = (S_1, \dots, S_p)$ of pairwise disjoint subsets of $\{1, \dots, k\}$ whose union is $\{1, \dots, k\}$ and with respective cardinalities $\lambda_1, \dots, \lambda_p$ (we have $\frac{k!}{\lambda_1! \cdots \lambda_p!}$ possibilities).

The corresponding collection $\mathbf{i} = (i_1, \dots, i_k)$ is then totally defined by the fact that for all ℓ , $i_\ell = i_r^0$, where $r \in \{1, \dots, p\}$ is such that $\ell \in S_r$. Note that in this case,

$$\# \mathfrak{S}_k^0(\mathbf{i}) \leq \lambda_1! \cdots \lambda_p!.$$

This gives

$$\mathbb{E} \operatorname{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) \leq \frac{Ck^2}{n^k} \sum_{p=1}^k \sum_{1 \leq i_1^0 < \dots < i_p^0 \leq n} \sum_{(\lambda_1, \dots, \lambda_p)} \sum_{\mathbf{S}} \prod_{\ell=1}^p s_{i_\ell}^{2\lambda_\ell} \prod_{\ell=1}^p \lambda_\ell!.$$

Then, note that

$$\prod_{\ell=1}^p s_{i_\ell}^{2\lambda_\ell} = \prod_{\ell=1}^p s_{i_\ell}^2 s_{i_\ell}^{2\lambda_\ell-2} \leq M^{2(k-p)} \prod_{\ell=1}^p s_{i_\ell}^2.$$

Hence changing the order of summation, we get

$$\begin{aligned}
\mathbb{E} \operatorname{Tr}(\mathbf{A}^k (\mathbf{A}^k)^*) &\leq \frac{Ck^2}{n^{k-1}} \sum_{p=1}^k M^{2(k-p)} \sum_{1 \leq i_1^0 < \dots < i_p^0 \leq n} \prod_{\ell=1}^p s_{i_\ell}^2 \sum_{(\lambda_1, \dots, \lambda_p)} \sum_{\mathbf{s}} \prod_{\ell=1}^p \lambda_\ell! \\
&\leq \frac{Ck^2}{n^{k-1}} \sum_{p=1}^k M^{2(k-p)} \frac{1}{p!} \sum_{1 \leq i_1^0, \dots, i_p^0 \leq n} \prod_{\ell=1}^p s_{i_\ell}^2 \sum_{(\lambda_1, \dots, \lambda_p)} k! \\
&\leq \frac{Ck^2}{n^{k-1}} \sum_{p=1}^k M^{2(k-p)} (nb^2)^p \frac{k!(k-1)!}{p!(p-1)!(k-p)!} \\
&\leq \frac{Ck^2}{n^k} \sum_{p=1}^k (kM^2)^{k-p} (nb^2)^p \frac{k!}{p!(k-p)!} \\
&\leq \frac{Ck^2}{n^{k-1}} (nb^2 + kM^2)^k \\
&\leq Cnk^2 \left(b^2 + \frac{kM^2}{n} \right)^k.
\end{aligned}$$

Let us now give the main lines of the proof of (5). This proof is very analogous to the one of (4), minus some border difficulties (we will use the symmetric group \mathfrak{S}_k instead of \mathfrak{S}_k^0).

Proceeding as above, we arrive easily at

$$\mathbb{E}[|\operatorname{Tr}(\mathbf{A}^k)|^2] = \sum_{\mathbf{i}=(i_1, \dots, i_k)} s_{i_1}^2 \cdots s_{i_k}^2 G_{\mathbf{i}}$$

with

$$G_{\mathbf{i}} := \sum_{\Phi \in \mathfrak{S}_k / \mathfrak{S}_k(\mathbf{i})} \mathbb{E}[u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_{k-1} i_k} u_{i_k i_1} \overline{u_{i_{\Phi(1)} i_{\Phi(2)}}} \overline{u_{i_{\Phi(2)} i_{\Phi(3)}}} \cdots \overline{u_{i_{\Phi(k-1)} i_{\Phi(k)}}} \overline{u_{i_{\Phi(k)} i_{\Phi(1)}}}],$$

where

$$\mathfrak{S}_k(\mathbf{i}) := \{\varphi \in \mathfrak{S}_k; \forall \ell, i_{\varphi(\ell)} = i_\ell\}$$

and $\mathfrak{S}_k / \mathfrak{S}_k(\mathbf{i})$ denotes the quotient set for the left action of $\mathfrak{S}_k(\mathbf{i})$ on \mathfrak{S}_k .

As above again, we get, for c the cycle $(1\ 2 \cdots k)$,

$$\begin{aligned}
G_{\mathbf{i}} &= \sum_{\Phi \in \mathfrak{S}_k / \mathfrak{S}_k(\mathbf{i})} \sum_{(\varphi_1, \varphi_2) \in \Phi \times \Phi} \operatorname{Wg}(c^{-1} \varphi_2^{-1} c \varphi_1) \\
&= \sum_{(\varphi, \alpha) \in \mathfrak{S}_k \times \mathfrak{S}_k(\mathbf{i})} \operatorname{Wg}(c^{-1} \varphi^{-1} \alpha^{-1} c \varphi).
\end{aligned}$$

Then applying Lemma 5, we get

$$|G_{\mathbf{i}}| \leq \sum_{\alpha \in \mathfrak{S}_k(\mathbf{i})} \left\{ \#\{\varphi \in \mathfrak{S}_k; c^{-1}\varphi^{-1}\alpha^{-1}c\varphi = id\} \left(\frac{1}{n^k} + \frac{k^2}{2n^{k+2}} \frac{1}{1 - \frac{k^4}{4n^2}} \right) + \frac{1}{1 - (k^2/(2n))} \sum_{q=1}^{k-1} \#\{\varphi \in \mathfrak{S}_k; |c^{-1}\varphi^{-1}\alpha^{-1}c\varphi| = q\} \frac{2}{n^k k^2} \left(\frac{k^2}{2n} \right)^q \right\}.$$

Then an analogue of Lemma 7 allows to claim that

$$|G_{\mathbf{i}}| \leq \frac{C}{n^k} \#\mathfrak{S}_k(\mathbf{i}),$$

and the end of the proof is quite analogous to what we saw above. \square

4. APPENDIX: WEINGARTEN CALCULUS

We recall this key-result about integration with respect to the Haar measure on the unitary group (see [7, Cor. 2.4] and [14, p. 61]). Let \mathfrak{S}_k denote the k th symmetric group.

Proposition 8. *Let k be a positive integer and $\mathbf{U} = [u_{ij}]_{i,j=1}^n$ a Haar-distributed matrix on the unitary group. Let $\mathbf{i} = (i_1, \dots, i_k, i'_1, \dots, i'_k)$, $\mathbf{j} = (j_1, \dots, j_k, j'_1, \dots, j'_k)$ be two $2k$ -uplets of $\{1, \dots, n\}$. Then*

$$\mathbb{E} [u_{i_1, j_1} \cdots u_{i_k, j_k} \overline{u_{i'_1, j'_1}} \cdots \overline{u_{i'_k, j'_k}}] = \sum_{\sigma, \tau \in \mathfrak{S}_k} \delta_{i_1, i'_{\sigma(1)}} \cdots \delta_{i_k, i'_{\sigma(k)}} \delta_{j_1, j'_{\tau(1)}} \cdots \delta_{j_k, j'_{\tau(k)}} \text{Wg}(\sigma^{-1}\tau), \quad (18)$$

where Wg is a function called the Weingarten function, depending implicitly on n and k and given by the formula

$$\text{Wg}(\pi) = \frac{1}{n^k} \sum_{r \geq 0} (-1)^r \frac{c_r(\pi)}{n^r}, \quad (19)$$

with

$$c_r(\pi) := \#\{(s_1, \dots, s_r, t_1, \dots, t_r); \forall i, 1 \leq s_i < t_i \leq k, t_1 \leq \dots \leq t_r, \pi = (s_1 t_1) \cdots (s_r t_r)\}.$$

Acknowledgments: We would like to thank J. Novak for discussions on Weingarten calculus and Camille Male for pointing out references [6] and [12] to us.

REFERENCES

- [1] A. Basak, A. Dembo *Limiting spectral distribution of sums of unitary and orthogonal matrices*. Electron. Commun. Probab. 18 (2013), no. 69, 19 pp.
- [2] F. Benaych-Georges, J. Rochet *Outliers in the Single Ring Theorem*, arXiv:1308.3064.
- [3] P. Bourgade, H.-T. Yau, J. Yin *The local circular law II: the edge case*, Probab. Theory Related Fields, Vol. 159, no. 3-4, 619–660 (2014).
- [4] D. Chafaï, S. Péché *A note on the second order universality at the edge of Coulomb gases on the plane*. J. Stat. Phys. 156 (2014), no. 2, 368–383.
- [5] B. Collins *Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability*. Int. Math. Res. Not. 2003, no. 17, 953–982.
- [6] B. Collins, C. E. González-Guillén, D. Pérez-García *Matrix Product States, Random Matrix Theory and the Principle of Maximum Entropy*, Comm. Math. Phys. 320 (2013), no. 3, 663–677.
- [7] B. Collins, P. Śniady *Integration with respect to the Haar measure on unitary, orthogonal and symplectic group*. Comm. Math. Phys., 264, 773–795, 2006.
- [8] A. Guionnet, M. Krishnapur, O. Zeitouni *The Single Ring Theorem*. Ann. of Math. (2) 174 (2011), no. 2, 1189–1217.
- [9] A. Guionnet, O. Zeitouni *Support convergence in the Single Ring Theorem*. Probab. Theory Related Fields 154 (2012), no. 3-4, 661–675.
- [10] U. Haagerup, F. Larsen *Brown’s spectral distribution measure for R -diagonal elements in finite von Neumann algebras*, Journ. Functional Analysis, 176 (2000), 331–367.
- [11] S. Matsumoto, J. Novak *Jucys-Murphy elements and unitary matrix integrals*. Int. Math. Res. Not. IMRN 2013, no. 2, 362–397.
- [12] A. Montanaro *Weak multiplicativity for random quantum channels*, Comm. Math. Phys., 319 (2013), no. 2, 535–555.
- [13] J. Novak *Jucys-Murphy elements and the unitary Weingarten function*. Noncommutative harmonic analysis with applications to probability II, 231–235, Banach Center Publ., 89, Polish Acad. Sci. Inst. Math., Warsaw, 2010.
- [14] J. Novak *Three lectures on free probability theory*. arXiv:1205.2097, to appear in MSRI Publications.
- [15] B. Rider *A limit theorem at the edge of a non-Hermitian random matrix ensemble*. J. Phys. A: Math. Gen. 36, 3401–3409 (2003).
- [16] B. Rider, C. Sinclair *Extremal laws for the real Ginibre ensemble*. Ann. Appl. Probab. 24 (2014), no. 4, 1621–1651.
- [17] M. Rudelson, *Lecture notes on non-aymptotic random matrix theory*, notes from the AMS Short Course on Random Matrices, 2013.
- [18] M. Rudelson, R. Vershynin *Non-asymptotic theory of random matrices: extreme singular values*, Proceedings of the International Congress of Mathematicians. Volume III, 1576–1602, Hindustan Book Agency, New Delhi, 2010.
- [19] M. Rudelson, R. Vershynin *Invertibility of random matrices: unitary and orthogonal perturbations*, J. Amer. Math. Soc. 27 (2014), 293–338.
- [20] I. Yin *The local circular law III: general case*. Probab. Theory Relat. Fields (2014) 160:679–732.

MAP 5, UMR CNRS 8145 - UNIVERSITÉ PARIS DESCARTES, 45 RUE DES SAINTS-PÈRES 75270 PARIS CEDEX 6, FRANCE.

E-mail address: florent.benaych-georges@parisdescartes.fr